

The systolic constant of orientable Bieberbach 3-manifolds

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Abstract

The *systole* of a compact non simply connected Riemannian manifold is the smallest length of a non-contractible closed curve ; the *systolic ratio* is the quotient $(\text{systole})^n/\text{volume}$. Its supremum, over the set of all Riemannian metrics, is known to be finite for a large class of manifolds, including the $K(\pi, 1)$.

We study the optimal systolic ratio of compact, 3-dimensional orientable Bieberbach manifolds which are not tori, and prove that it cannot be realized by a flat metric.

Key words and phrases. Systole; systolic ratio; singular Riemannian metric; Bieberbach manifold.

1 Introduction

1.1 Motivations and main result

The systole of a compact non simply connected Riemannian manifold (M^n, g) is the shortest length of a non contractible closed curve, we denote it by $\text{Sys}(g)$. To get an homogeneous Riemannian invariant, we introduce the *systolic ratio* $\frac{\text{Sys}(g)^n}{\text{Vol}(g)}$. It is important to note that this invariant is well defined even if g is only continuous, i.e. a continuous section of the fiber bundle S^2T^*M of symmetric forms.

An isosystolic inequality on a manifold M is a inequality of the form

$$\frac{\text{Sys}(g)^n}{\text{Vol}(g)} \leq C$$

that holds for any Riemannian metric g on M . The smallest such constant C is called the *systolic constant*.

A systolic geodesic will be for us a closed curve, not homotopically trivial, whose length is equal to the systole.

In 1949, in an unpublished work, C. Loewner proved the following result. For any metric h on the torus \mathbb{T}^2 we have

$$\frac{\text{sys}^2(\mathbb{T}^2, h)}{\text{Area}(\mathbb{T}^2, h)} \leq 2/\sqrt{3}$$

Furthermore, the equality is achieved if and only if (\mathbb{T}^2, h) is isometric to a flat hexagonal torus.

In 1952, P.M. Pu proved an isosystolic inequality for the real projective plane (c.f. [Pu52]). The extremal metric has constant curvature, too. In the same paper, he proved a variant of the isosystolic inequality for the Möbius bands with boundary, valid for each conformal class of any metric.

There exists a third case, solved by C. Bavard in [Bav86], where the upper bound of the systolic ratio is known, and realized, this is the case of the Klein bottle. This time the extremal metric (for the isosystolic inequality) is singular, more precisely piecewise C^1 (see [Bav86]). Furthermore, it has curvature $+1$ where it is smooth.

In higher dimension, there exists non simply connected manifolds that do not satisfy any isosystolic inequality. The simplest example is $S^2 \times S^1$, or more generally the product of a simply connected manifolds by a non simply connected one. Making the volume of the simply connected factor tend to zero insures the explosion of the systolic ratio..

However a fundamental result of M. Gromov (cf. [Gro83]), ensures that *essential manifolds* satisfy an isosystolic inequality. A compact manifold M is *essential* if there exists a continuous map from M in a $K(\pi, 1)$ ($\pi = \pi_1(M)$) which sends the fundamental class to a non trivial one. The essential manifolds include notably aspherical manifolds and the real projectives. Furthermore, I. Babenko proved in [Bab92] a reciprocal of the theorem of M. Gromov : "In the orientable case, essential manifolds are the only manifolds that satisfy isosystolic inequalities".

However, in dimensions ≥ 3 , hardly anything is known about metrics that realize the systolic constant (extremal metrics). It is not known for example, in the apparently simple cases of tori and real projective spaces, whether the metrics of constant curvature are extremal.

In the present work, we are interested in *Bieberbach manifolds*, i.e. compact manifolds that carry a flat Riemannian metric. These manifolds are $K(\pi, 1)$, and then the theorem of Gromov can be applied. Our result is the following

Let M be a Bieberbach orientable manifold of dimension 3 that is not a torus. Then there exists on M a Riemannian metric g such that, for any flat metric h ,

$$\frac{(\text{sys}(h))^3}{\text{vol}(h)} < \frac{(\text{sys}(g))^3}{\text{vol}(g)}$$

We recently proved the same result for non-orientable 3-dimensional Bieberbach manifolds (see [El-La08]). The main idea consisted in the fact that we can get any such manifold by suspending a flat Klein bottle. Putting then the spherical metric of Bavard on these Klein bottles give metrics (locally isometric to $S^2 \times \mathbb{R}$) whose systolic ratio is better than the flat ones.

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1.2 Idea of the proof

The proof relies on the fact that these manifolds contain an isolated systolic geodesic on one hand, and a lot of surfaces that are flat Klein bottles and flat Möbius bands (except for C_3 but this case can be treated similarly) on the other. To see this we use a theorem of classification of flat manifolds of dimension 3, this theorem is a result of the theorem of Bieberbach for crystallographic groups (see [Wol74] and [Cha86]).

In section 3 we define in the setting of Riemannian polyhedrons the Riemannian singular spaces. The extremal Klein bottle (for the isosystolic inequality) fits into this setting. We introduce then Riemannian singular metrics (with the preceding meaning) on the orientable Bieberbach manifolds of dimension 3 of type C_2, C_3, C_4 and C_6 . (We follow the notations of W. Thurston, see [Thu97])

For these metrics the Klein bottles and Möbius bands become singular surfaces of curvature $+1$ outside the singularity. It is useful to note that the group of isometries of these metrics is the same as for the flat ones. The case of the manifold $C_{2,2}$ is treated thanks to the following (probably folk) result communicated to us by I. Babenko.

if g is an extremal Riemannian metric (eventually singular) on M , the systolic geodesics do cover M (see [Cal96]).

This property is satisfied by flat tori, and real projectives endowed with their metric of constant curvature. On Bieberbach manifolds of dimension 3, the metrics that optimize the systolic ratio *among flat metrics* also satisfy this property, except for the manifold $C_{2,2}$ (cf. [Wol74] p.117-118, and the suggestive figure of [Thu97], p.236). This property gives the result for $C_{2,2}$.

The metrics that we construct also satisfy this property, and so there is no obvious obstacle that prevents them from being extremal (especially the one on C_2 , see section 7).

2 Flat manifolds

2.1 Classification of flat manifolds

Compact flat manifolds are quotients \mathbb{R}^n/Γ , where Γ is a discrete cocompact subgroup of affine isometries of \mathbb{R}^n acting freely. By the theorem of Bieberbach Γ is an extension of a finite group G by a lattice Λ of \mathbb{R}^n . This lattice is the subgroup of the elements of Γ that are translations, we obtain then the following exact sequence:

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

Actually, if M is a flat manifold, M is the quotient of the flat torus \mathbb{R}^n/Λ by an isometry group

isomorphic to G . Two compact and flat manifolds \mathbb{R}^n/Γ and \mathbb{R}^n/Γ' are homeomorphic if and only if the groups Γ and Γ' are isomorphic. Such groups are then conjugate by an affine isomorphism of \mathbb{R}^n : two compact and flat homeomorphic manifolds are affinely diffeomorphic.

2.2 3-dimensional orientable flat manifolds

The classification of flat manifolds of dimension 3 results of a direct method of classification of discrete, cocompact subgroups of $\text{Isom}(\mathbb{R}^3)$ operating freely. This classification is due to W. Hantzsche and H. Wendt (1935), and exposed in the book [Wol74] of J.A. Wolf. There exists ten compact and flat manifolds of dimension 3 up to diffeomorphism, six are orientable and four are not.

In the orientable case, these types are characterized by the holonomy group G , this reason motivates our notation. A rotation of angle α around an axis a will be denoted by $r_{a,\alpha}$.

- i) $G = \{1\}$: type C_1 . This is the torus, it is the quotient of \mathbb{R}^3 by an arbitrary lattice of \mathbb{R}^3 .
- ii) $G = \mathbb{Z}_2$: type C_2 . Given a basis (a_1, a_2, a_3) of \mathbb{R}^3 with $a_3 \perp (a_1, a_2)$, let Γ be the subgroup of isometries of \mathbb{R}^3 generated by $t_{a_3/2} \circ r_{a_3,\pi}$ and the translations t_{a_1} and t_{a_2} . The quotient \mathbb{R}^3/Γ is a manifold of type C_2 . Note that the lattice Λ generated by t_{a_1} , t_{a_2} et t_{a_3} is of index 2 in Γ .

This manifold is the quotient of the torus \mathbb{R}^3/Λ by the cyclic group of order 2 generated by (the image of) $t_{a_3/2} \circ r_{a_3,\pi}$.

- iii) $G = \mathbb{Z}_4$: type C_4 . Given an orthogonal basis (a_1, a_2, a_3) of \mathbb{R}^3 with $|a_1| = |a_2|$, let Γ be the subgroup of isometries of \mathbb{R}^3 generated by $t_{a_3/4} \circ r_{a_3,\pi/2}$ and the translations t_{a_1} et t_{a_2} . The quotient \mathbb{R}^3/Γ is a manifold of type C_4 . Note that the lattice Λ generated by t_{a_1}, t_{a_2} and t_{a_3} is of index 4 in Γ .

This manifold is the quotient of the torus \mathbb{R}^3/Λ by the cyclic group of order 4 generated by (the image of) $t_{a_3/4} \circ r_{a_3,\pi/2}$. It is also the quotient of C_2 (the basis (a_1, a_2, a_3) should be chosen orthogonal with $|a_1| = |a_2|$), by the subgroup generated by $t_{a_3/4} \circ r_{a_3,\pi/2}$. This remark will play an important role in the improvement of the systolic ratio of C_4 .

- iv) $G = \mathbb{Z}_6$: type C_6 . Given a basis (a_1, a_2, a_3) of \mathbb{R}^3 with $a_3 \perp (a_1, a_2)$, $|a_1| = |a_2|$ and $(a_1, a_2) = \pi/3$, let Γ be the subgroup of isometries of \mathbb{R}^3 generated by $t_{a_3/6} \circ r_{a_3,\pi/3}$ and the translations t_{a_1} et t_{a_2} . The quotient \mathbb{R}^3/Γ is a manifold of type C_6 . This time, the lattice Λ generated by t_{a_1}, t_{a_2} and t_{a_3} is of index 6 in Γ .

C_6 is the quotient of the torus \mathbb{R}^3/Λ by the cyclic group of order 6 generated by (the image of) $t_{a_3/6} \circ r_{a_3,\pi/3}$. It is also the quotient of C_2 by the subgroup generated by $t_{a_3/6} \circ r_{a_3,\pi/3}$.

- v) $G = \mathbb{Z}_3$: type C_3 . Given a basis (a_1, a_2, a_3) of \mathbb{R}^3 with $a_3 \perp (a_1, a_2)$, $|a_1| = |a_2|$ and $(a_1, a_2) = 2\pi/3$, let Γ be the subgroup of isometries of \mathbb{R}^3 generated by $t_{a_3/3} \circ r_{a_3,2\pi/3}$ and the

translations t_{a_1} et t_{a_2} . The quotient \mathbb{R}^3/Γ is a manifold of type C_3 . The lattice Λ generated by t_{a_1}, t_{a_2} and t_{a_3} is of index 3 in Γ . This manifold is the quotient of the torus \mathbb{R}^3/Λ but it is not a quotient of C_2 .

vi) $G = \mathbb{Z}_2 \times \mathbb{Z}_2$: type $C_{2,2}$. Given an orthogonal basis (a_1, a_2, a_3) of \mathbb{R}^3 , let Γ be the subgroup of isometries of \mathbb{R}^3 generated by $t_{a_1/2} \circ r_{a_1, \pi}$, $t_{(\frac{a_1+a_2}{2})} \circ r_{a_2, \pi}$ and $t_{(\frac{a_1+a_2+a_3}{2})} \circ r_{a_3, \pi}$. The quotient \mathbb{R}^3/Γ is the manifold $C_{2,2}$. This time, the holonomy group G is not cyclic, it is equal to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Remark 1. In the case of the manifolds C_2 , C_4 and C_6 , every plane that contains a_3 gives, in the quotient, a flat Klein bottle (if the plane contains points of the lattice other than those of the axis a_3) or a flat Möbius band without boundary (otherwise).

3 Singular metrics on Bieberbach manifolds

All the singular metrics we will use give rise to length spaces, i.e. spaces with a notion of length for simple closed curves. In this section we will define these metrics in a general case, the Riemannian polyhedrons. For further details on this notion see [Bab02].

A polyhedron is a topological space endowed with a triangulation, i.e. divided into simplexes glued together by their faces. We denote by σ an arbitrary simplexe of a polyhedron P .

Definition 1. A Riemannian metric on a polyhedron P is a family of Riemannian metrics $\{g_\sigma\}_{\sigma \in I}$, where I is in bijection with the set of simplexes of P . These metrics should satisfy the following conditions:

1. Every g_σ is a smooth metric in the interior of the simplex σ .
2. The metrics g_σ coincide on the faces; i.e. for any pair of simplexes σ_1, σ_2 , we have the equality

$$g_{\sigma_1}|_{\sigma_1 \cap \sigma_2} = g_{\sigma_2}|_{\sigma_1 \cap \sigma_2}$$

Such a Riemannian structure on the polyhedron allows us to calculate the length of any piecewise smooth curve in P , this way, the polyhedron (P, g) turns out to be a length space. If γ is a piecewise smooth path from an interval I to P , then the length of γ is defined as for the C^∞ metrics:

$$l(\gamma) = \int_I (g(\gamma'(t), \gamma'(t)))^{1/2} dt.$$

Furthermore (P, g) gains a structure of metric space (and especially a structure of length space) the same way as for smooth manifolds.

$$d_g(x, y) = \inf_{\gamma} l(\gamma)$$

where γ runs the set of piecewise smooth paths from x to y .

It is useful to note that the Riemannian measure too, is defined exactly as in the smooth case. Of course, the volume of the singular part will be zero.

The geodesics of a Riemannian polyhedron are the geodesics of the associated length structure (see [Bu-Iv01]). In the interior of a simplex (σ, g_σ) , the first variation formula shows that such a geodesic is a geodesic of g_σ in the Riemannian sense.

3.1 The Klein-Bavard bottle

The flat Klein bottle are the manifolds \mathbf{R}^2/Γ , where Γ is the subgroup of isometries of \mathbf{R}^2 generated by the glide reflection $(x, y) \mapsto (x + \frac{a}{2}, -y)$ and the translation $(x, y) \mapsto (x, y + b)$. We know by Bavard ([Bav86]) that any flat Klein bottle is not be extremal for the isosystolic inequality (see also [Gro83]), the unique extremal one is singular and has curvature $+1$ outside the singularities:

We start with the round sphere, and we locate the points by their latitude ϕ and their longitude θ . For $\phi_o \in]0, \pi/2[$, let Σ_{ϕ_o} be the domain defined by $|\phi| \leq \phi_o$. In Σ_{ϕ_o} , the round metric is given by $d\phi^2 + \cos^2 \phi d\theta^2$. Note that the universal cover of Σ_{ϕ_o} is the strip $\mathbf{R} \times [-\phi_o, \phi_o]$ with the same metric ($d\phi^2 + \cos^2 \phi d\theta^2$). Here we introduce in \mathbf{R}^2 the singular Riemannian metric

$$d\phi^2 + f^2(\phi)d\theta^2, \tag{1}$$

where f is the $2\phi_o$ periodic function which agrees with $\cos \phi$ in the interval $[-\phi_o, \phi_o]$.

Example 1. The metric on the Klein bottle that gives the best systolic ratio ($\frac{\pi}{2\sqrt{2}}$) is obtained for $\phi_o = \frac{\pi}{4}$ by taking the quotient of the plane endowed with the metric 1 by the action of the group generated by

$$(\theta, \phi) \mapsto (\theta + \pi, -\phi) \quad \text{et} \quad (\theta, \phi) \mapsto (\theta, \phi + 4\phi_o).$$

For more details on the Klein-Bavard bottle (that we denote (\mathbf{K}, b)) see [El-La08] and [Bav88].

Remark 2. It may seem more natural to take the quotient of the plane (endowed with the metric 1) by the group generated by

$$(\theta, \phi) \mapsto (\theta + \pi, -\phi) \quad \text{and} \quad (\theta, \phi) \mapsto (\theta, \phi + 2\phi_o)$$

the surface we obtain is indeed a Klein bottle but it does not give the best systolic ratio: the geodesics closed by the correspondence $(\theta, \phi) \mapsto (\theta, \phi + 2\phi_o)$ have a length equal to $\pi/2$ whereas the ones closed by the correspondence $(\theta, \phi) \mapsto (\theta, \phi + \phi_o)$ have length π . It is then possible to reduce the volume without shortening the systole by reducing the metric in the direction of the long closed curves.

3.2 Singular metrics on orientable Bieberbach manifolds

Starting with an arbitrary lattice of Δ of \mathbb{R}^2 , we introduce the associated Dirichlet-Voronoi paving. It is a paving of the plane by hexagons (or rectangles if the lattice Δ is rectangle) A_p centered at the points p of the lattice. Then a lattice of \mathbb{R}^3 of the form $\Delta \times c\mathbb{Z}$, where $c \in \mathbb{R}$, allows us to pave \mathbb{R}^3 naturally with hexagonal or rectangular prisms that we denote by D_p .

Now we introduce on \mathbb{R}^3 the Riemannian singular metric $h = dx^2 + dy^2 + \psi(m)dz^2$, where we set, for $m(x, y, z) \in D_p$, $\psi(m) = \cos^2 \text{dist}((x, y), p)$, with $\text{dist}((x, y), p) < \pi/2$. If m is in two domains D_p and $D_{p'}$ then p and p' are at the same distance from m : the map ψ is well defined. It is continuous, but it is not C^1 . The connected component of the identity in the group of isometries of (\mathbb{R}^3, h) consists of the vertical translations $(x, y, z) \mapsto (x, y, z + c')$. The translations by the vectors of Δ are also isometries. It is important to note that the metric h can also be written in the form $dx^2 + dy^2 + \cos^2(d((x, y), \Delta))dz^2$, where $d((x, y), \Delta)$ is the distance from the point (x, y) to the lattice Δ .

The quotient of (\mathbb{R}^3, h) by the group $\Delta \times c\mathbb{Z}$ (where $c > 0$) is a singular torus of dimension 3. We denote by (T, h) this special torus. The sections of (T, h) by the planes $z = \text{constant}$ are 2-dimensional totally geodesic flat tori. All these flat tori are isometric to \mathbb{R}^2/Δ . Note that the map from (T, h) into the torus \mathbb{R}^2/Δ , which consists in projecting onto the torus $z = 0$, is a Riemannian submersion.

With a good choice of the lattice Δ , the transformations $t_{a_3/n} \circ r_{a_3, 2\pi/n}$ ($n = 2, 3, 4, 6$), described in the classification of the flat orientable manifolds, become isometries of (T, h) (The lattice Δ should be square to get C_4 and hexagonal to get C_3 et C_6). This way we get a family of singular Riemannian metrics on the manifolds of type C_2, C_3, C_4 and C_6 .

Remark 3. Actually this construction works if we take the quotient of (\mathbb{R}^3, h) by the lattice $n\Delta \times c\mathbb{Z}$. Starting with this torus, we can re-obtain all the manifolds C_i ($i = 2, 3, 4, 6$) exactly the same way as for the torus (T, h) . We will see in the next sections that taking $n = 2$ is more useful to get "good systolic ratios" on these manifolds.

4 Two singular tori and their systole

We take the quotient of the Riemannian singular space (\mathbb{R}^3, h) seen in section 3.2 by the lattice $2\Delta \times 2\pi\mathbb{Z}$. We get a 3-dimensional torus (T, g) whose singular area is connected. It consists of the boundary of four hexagonal (or rectangular) prisms constituting a fundamental domain for the action of $2\Delta \times c\mathbb{Z}$. If u and v are two vectors generating the lattice Δ then the sections of (T, g) by planes containing a point of the lattice Δ and parallel to u (or v) are tori of dimension 2 and of curvature +1 outside their singular area. Taking the good choice of u and v these tori are the orientable covering of the Klein-Bavard bottle (\mathbf{K}, b) introduced in [Bav86]. In general, the sections of (T, g) by planes containing the axis of a domain D_p are surfaces of curvature +1 as long as we stay in the interior of D_p . We denote these surfaces by S_p .

Remark 4. To preserve the systole and reduce the volume of the manifolds of type C_i it is

crucial to take the quotient of (\mathbb{R}^3, h) by the lattice $2\Delta \times 2\pi\mathbb{Z}$ and not by $\Delta \times 2\pi\mathbb{Z}$. This prevents shortening closed curves at the level of the surfaces S_p .

First suppose that the lattice Δ is square and generated by two vectors of norm $2a > 0$. This lattice is generated by three translations $t_1 : (x, y, z) \longrightarrow (x + 4a, y, z)$, $t_2 : (x, y, z) \longrightarrow (x, y + 4a, z)$ and $t_3 : (x, y, z) \longrightarrow (x, y, z + 2\pi)$. We denote by (T, g_c) the quotient torus. Its singular area consists of four connected surfaces $x = a$, $x = 3a$, $y = a$ and $y = 3a$.

Note that the symmetries with respect to the surfaces $x = pa$ and $y = qa$ where $p, q \in \mathbb{Z}$, are isometries of (T, g_c) .

Lemma 1. *The systole of (T, g_c) is equal to $\inf\{4a, 2\pi \cos(a\sqrt{2})\}$.*

Proof. Let γ be a curve in (\mathbb{R}^3, h) , from $m(x_0, y_0, z_0)$ to $t_1(m)$, then

$$l(\gamma) \geq \int \sqrt{x'^2 + y'^2 + \psi(x, y)z'^2} dt \geq \int x' dt \geq 4a$$

Now, if γ is a curve from $m(x_0, y_0, z_0)$ to $t_2(m)$ we find exactly the same way as before that $l(\gamma) \geq 4a$, just compare the length of γ to its projection on $\{y = y_0, z = z_0\}$. Finally, for a curve γ from m to $t_3(m)$, we have

$$l(\gamma) = \int \sqrt{x'^2 + y'^2 + \psi(x, y)z'^2} dt \geq \int \inf(\psi) z' dt = 2\pi \cos a\sqrt{2}$$

the equality is obtained for the points of the edges of the square prism D_p . Using exactly the same technique we can prove that the distance between a point m and its image by the composition of several translations is greater or equal to $\inf\{4a, 2\pi \cos a\sqrt{2}\}$.

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Suppose now that the lattice Δ is hexagonal and generated by two vectors of norm $2a > 0$. The lattice $2\Delta \times 2\pi\mathbb{Z}$ is generated by the translations $T_1 : (x, y, z) \longrightarrow (x + 4a, y, z)$, $T_2 : (x, y, z) \longrightarrow (x + 2a, y + 2a\sqrt{3}, z)$ and $T_3 : (x, y, z) \longrightarrow (x, y, z + 2\pi)$. The manifold we get is a singular torus that we denote by (T, g_{hex}) . Its singular area consists of the edges of the hexagonal prisms D_p that pave \mathbb{R}^3 .

Remark 5. The symetries with respect to the surfaces $x = pa$, $y + \frac{x}{\sqrt{3}} = \frac{2pa}{\sqrt{3}}$ and $y - \frac{x}{\sqrt{3}} = \frac{2pa}{\sqrt{3}}$, are isometries of (T^3, g_{hex}) .

Lemma 2. *The systole of (T, g_{hex}) is equal to $\inf\{4a, 2\pi \cos(2a/\sqrt{3})\}$.*

Proof. For any curve γ from m to $T_1(m)$ we have

$$l(\gamma) \geq \int \sqrt{x'^2 + y'^2 + \psi(x, y)z'^2} dt \geq \int x' dt \geq 4a$$

the same inequality holds for any curve from m to $T_2(m)$ since the situation is invariant by the rotation $r_{a_3, \pi/3}$ of angle $\pi/3$ around the axis z .

Finally, for any curve γ from m to $T_3(m)$, we have

$$l(\gamma) = \int \sqrt{x'^2 + y'^2 + \psi(x, y)z'^2} dt \geq \int \inf(\psi) z' dt = 2\pi \cos(2a\sqrt{3})$$

the equality is achieved for the points of the edges of the hexagonal prisms D_p . The distance between a point m and its image by the composition of several translations is greater or equal to $\inf\{4a, 2\pi \cos(2a\sqrt{3})\}$.

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5 The systolic ratio of C_2

5.1 The systolic ratio in the case of flat metrics

The volume is equal to $\frac{1}{2} \det(a_2, a_1) |a_3|$ and the systole is equal to $\inf\{|a_3|/2, s\}$, where s is the systole of the flat torus of dimension 2 defined by the lattice of basis a_1, a_2 . We normalize such that $\frac{1}{2}|a_3| = 1$, then the systolic ratio is equal to

$$\frac{s^3}{\det(a_1, a_2)} \quad \text{if } s \leq 1 \quad \text{and} \quad \frac{1}{\det(a_1, a_2)} \quad \text{if } s \geq 1,$$

Since we have $\frac{s^2}{\det(a_1, a_2)} \leq \frac{2}{\sqrt{3}}$ (lattice of dimension 2), the systolic ratio is less or equal to $2/\sqrt{3}$.

5.2 Klein bottles and Möbius bands in C_2

We have already seen that the planes containing a_3 give rise to flat Klein bottles or flat Möbius bands without boundary. If the plane contains a point of Λ (c.f. 2.2) other than those of the axis a_3 , the intersection is a Klein bottle, otherwise it is a Möbius band.

To improve the systolic ratio $2/\sqrt{3}$, we should reduce the volume without touching the systole. This can be done by taking benefit of the non orientable surfaces embedded in the flat manifold C_2 and "put" the spherical metric of Bavard and Pu on them.

5.3 A singular metric on C_2 better than the flat ones

We start with the singular torus (T, g_{hex}) seen in section 4.

The transformation $\sigma : (x, y, z) \longrightarrow (-x, -y, z + \pi)$ is an isometry of the metric g . To get a manifold homeomorphic to C_2 we must take the quotient of (T, g_{hex}) by the subgroup generated by σ . We denote this manifold by (C_2, g_{hex}) .

In the torus (T, g_{hex}) , the transformation σ keeps 4 geodesics globally invariant, these are the vertical axes containing the 4 centers of the prisms that constitute a fundamental domain of C_2 (this is the set $\{x = 2pa, y = 2qa, (p, q) \in \mathbb{Z}^2\}$). Actually, this is a property of the fundamental group of C_2 that does not depend on the metric and holds for any metric on C_2 . The existence of these geodesics is a bit disturbing since they can shorten some closed curves in the manifold (C_2, g_{hex}) .

We go back now to the metric g , it can be written locally (in the domain D) in cylindrical coordinates (with respect to x and y) in the form $g = dr^2 + r^2 d\theta^2 + \cos^2 r dz^2$ (r is the distance to the vertical axes going through the center p of the prism D_p , and θ is the angle with respect to the axis "x"). In the following, we will even consider the first form or the other depending what we need.

Remark 6. In restriction to a prism D_p , a surface of equation $\theta = \theta_0$ is totally geodesic. To see that, just remark that the length of any curve γ in D_p joining two points of $\theta = \theta_0$ is always greater than its projection on this surface. This is simply due to the expression of the metric in the "cylindrical" coordinates:

$$l(\gamma) = \int \sqrt{r'^2 + r^2 \theta'^2 + \cos^2 r z'^2} dt \geq \int \sqrt{r'^2 + \cos^2 r z'^2}$$

The surfaces $\theta = \text{constant}$ are not singular as long as we stay in the interior of a domain D_p , they are locally isometric to S^2 and their geodesics are also geodesics of (C_2, g_{hex}) .

Lemma 3. *Let γ be a curve of the universal Riemannian covering of (T, g_{hex}) and γ' its minimal projection on a hexagonal prism D_p , then we have $l(\gamma) \geq l(\gamma')$.*

Proof. The *minimal projection* of a point m is here the point of D_p at a minimal distance (Euclidian) of m . It is unique since D_p is convex.

If the minimal orthogonal projection is completely inside the singularity $x = pa$, i.e. if γ' is in such a hypersurface, then

$$l(\gamma) = \int \sqrt{x'^2 + y'^2 + \psi(x, y) z'^2} dt \geq \int \sqrt{y'^2 + \psi(x, y) z'^2} dt = l(\gamma')$$

but the situation is invariant by a rotation of angle $\pi/3$ around p ; this shows that if γ is projected on the surfaces $y + \frac{x}{\sqrt{3}} = \frac{2pa}{\sqrt{3}}$ or $y - \frac{x}{\sqrt{3}} = \frac{2pa}{\sqrt{3}}$ of the singularity, we have $l(\gamma) \geq l(\gamma')$. Finally, the result is true for any curve projected on anywhere on the singularity. ■

Remark 7. In fact the previous lemma holds even if we take the minimal projection on a hexagonal prism *inside* D_p and parallel to it. A prism is parallel to D_p if every plane consisting its boundary is parallel to a plane of the boundary of D_p . This remark will be used in the improvement of the systolic ratio of the manifold C_3 .

Lemma 4. *For any point $m(r_0, \theta_0, z_0)$ in (T, g_{hex}) we have*

$$d_{(T, g_{hex})}(m, \sigma(m)) \geq \pi.$$

The equality is achieved for a geodesic of the surface $\theta = \theta_0$.

Proof. Let $m(r_0, \theta_0, z_0)$ be a point in D_p , and γ a curve in (\mathbb{R}^3, h) from m to $\sigma(m)$. If γ stays in D_p , then by Remark 6 we have $l(\gamma) \geq l(pr(\gamma))$ where $pr(\gamma)$ is the projection of γ on the surface $\theta = \theta_0$. But $l(pr(\gamma)) \geq \pi$ since the metric on this surface is spherical ($dr^2 + \cos^2 r d\theta^2$). Now if γ

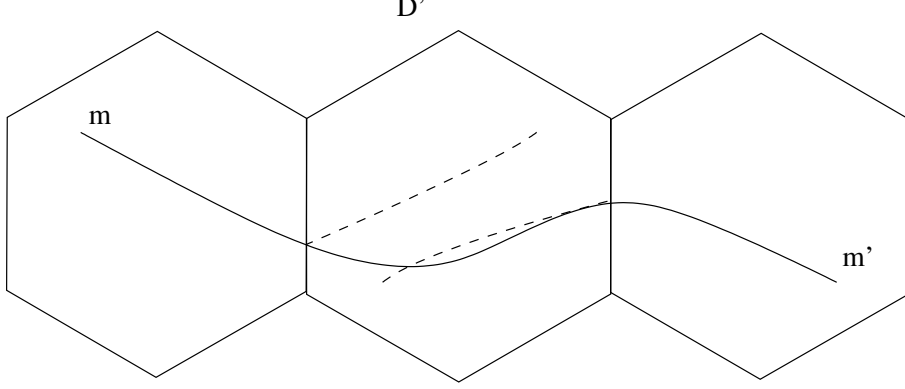


Figure 1: A curve joining m to $m' = T_1(\sigma(m))$ will go through 3 domains D_p . For the parts of this curve outside D' we take their symetrics with respect to the boundary D' .

gets out of the prism D_p , let γ' be the curve obtained by taking the projection (minimal) of the part of γ outside D_p on the boundary ∂D_p , and by leaving the part inside D_p unchanged. Then γ' is a curve of D_p from m to $\sigma(m)$. Its length is greater or equal to π (using the same argument of projection on the surface $\theta = \theta_0$). We conclude that $l(\gamma) \geq l(\gamma')$.

Then we have to calculate in (\mathbb{R}^3, h) (a lower bound of) the distance to (a lift of) $\sigma(m)$ of the images of m by translations. We denote by σ_0 any lift of σ in (\mathbb{R}^3, h) . If we translate m by T_3 , the situation will be equivalent to the one above since $T_3(m)$ and $\sigma_0(m)$ are conjugate by σ_0^{-1} . Now a curve γ in (\mathbb{R}^3, h) from $\sigma_0(m)$ to $T_1(m)$ should go through at least 3 domains D_p . Among these let D' be the domain that neither contains $\sigma_0(m)$ nor $T_1(m)$.

- If γ stays in these three domains, let γ' be the curve obtained by taking symetrics of the parts of γ outside D' with respect to the singular "plane" of $\partial D'$ beside the curve (see fig.3.1). The curve γ' is in D' , it joins two conjugate points by the transformation σ_0 , then $l(\gamma) \geq l(\gamma') \geq \pi$ (above argument).
- If γ gets out of these domains, let γ' be the curve obtained by projecting the part of γ outside D' on its boundary $\partial D'$. We get a continuous curve in D' joining two conjugate points by σ_0 , we conclude that $l(\gamma) \geq l(\gamma') \geq \pi$.

Finally, note that the distance to $\sigma_0(m)$ of the composition of several translations of m is too large by arguments similar to those above. ■

Remark 8. In fact the two preceding lemmas are also true for the torus (T, g_c) and can be proven exactly the same way.

Theorem 1. *If the real number a is equal to $\pi/4$ then*

$$\frac{Sys^3(C_2, g_{hex})}{Vol(C_2, g_{hex})} > \frac{2}{\sqrt{3}}$$

Proof. The volume of (C_2, g_{hex}) is equal to

$$\int_0^\pi \iint_D \cos \sqrt{x^2 + y^2} dy dx dz$$

where D is a regular hexagon of shortest distance between its opposite edges equal to $2a$.

The systole is equal to

$$\inf \{ Sys(T, g_{hex}), \inf \{ dist_{(T, g_{hex})}(m, \sigma(m)) \} \}$$

By Lemmas 2 and 4, it is equal to $\inf \{ 4a, 2\pi \cos(a\sqrt{2}), \pi \}$. Then, for $a = \pi/4$, we have $Sys(C_2, g_{hex}) = \pi$. Using the software "Maple" we find an approximation of the volume (2,80) up to 1/100, then a simple calculation gives the systolic ratio $Sys^3(C_2, g_{hex})/Vol(C_2, g_{hex}) \simeq 1,38$. ■

5.4 The manifold C_2 as a quotient of the torus (T, g_c)

To get a manifold homeomorphic to C_2 , we can take an arbitrary Δ , then consider the quotient by the same transformations as before. To increase the most the systolic ratio, Δ should have the smallest volume possible, i.e. it should be hexagonal. It is nevertheless interesting to get this manifold as a quotient of the torus (T, g_c) , i.e. when Δ is the "special" square lattice. We denote by (C_2, g_c) the quotient of (T, g_c) by the subgroup generated by σ .

When $a = \pi/4$, the intersection of (T, g_c) with one of the planes $x = 0$ or $y = 0$ is the covering torus of the Klein-Bavard bottle (c.f. [Bav86], see also [El-La08]). More generally, the intersection with planes containing the axis z is a singular surface (a cylinder or a torus) of curvature $+1$ where it is smooth. It turns out to be true that with a good choice of the parameter a the manifold (C_2, g_c) admits a systolic ratio greater than $\sqrt{3}/2$, and the calculation is based, as in the case of (T, g_{hex}) , on the fact that the distance in (T, g_c) between a point and its image by σ is greater than π (c.f. 4).

Proposition 1. *If the real number a satisfies the equation $2a - \pi \cos a\sqrt{2} = 0$, then the systolic ratio $\frac{Sys^3(C_2, g_c)}{Vol(C_2, g_c)}$ is greater than $2/\sqrt{3}$. It is approximately equal to 1,18.*

6 The systolic ratio of C_4 , C_6 , C_3 and $C_{2,2}$

6.1 Type C_4

In the flat case, we saw that C_4 is the quotient of C_2 (the basis (a_1, a_2, a_3) should be orthogonal with $|a_1| = |a_2|$) by the subgroup generated by $t_{a_3/4} \circ r_{a_3, \pi/2}$. It turns out that this property is true for the metric g_c , more precisely the transformation $t_{a_3/4} \circ r_{a_3, \pi/2}$ is indeed an isometry of g_c and the quotient of (C_2, g_c) by this isometry gives a manifold of type C_4 .

The volume of a flat manifold of type C_4 is equal to $|a_1||a_2||a_3|/4$, and the systole is equal to $\inf\{|a_1|, |a_2|, |a_3|/4\}$. The systolic ratio is smaller than 1.

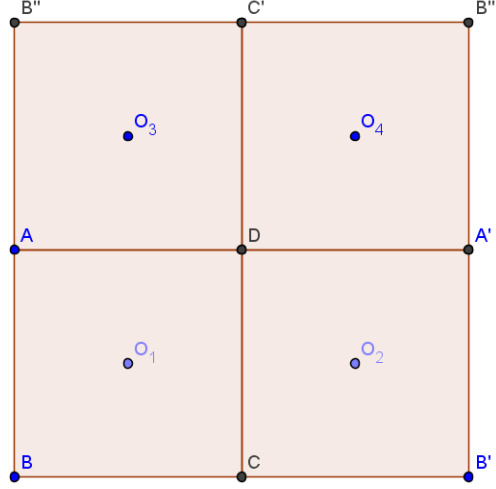


Figure 2: The transformations τ et τ^{-1} keep fixed the vertical axes going through the points O_1 et O_4 . The transformation τ^2 keep fixed these same axes, as well as the vertical ones going through the points O_3 and O_4 .

Now, the quotient of (T, g_c) by the subgroup Γ of isometries of g generated by $\tau : (x, y, z) \longrightarrow (-y, x, z + \pi/2)$ gives a manifold homeomorphic to C_4 , we denote it by (C_4, g_c) . Actually C_4 can be seen in two different ways starting from the isometry $t_{a_3/4} \circ r_{a_3, \pi/2}$ (which is the same as τ) of \mathbb{R}^3 . This isometry gives when we go to the quotient a fixed points free isometry of order 4 (resp of order 2) of (T, g_c) (resp of (C_2, g_c)).

The transformations τ and τ^{-1} are of order 4 in (T, g_c) and keep 2 geodesics globally invariant.

The transformation τ^2 is of order 2 in (T, g_c) and keep, in addition to the geodesics fixed by the transformation τ , 2 others globally invariant. They are the vertical geodesics going through the points of the lattice Δ (see fig.2).

Theorem 2. *If $a = \pi/8$, the systole of (C_4, g_c) is equal to $\pi/2$ and the systolic ratio $\frac{Sys^3(C_4, g_c)}{Vol(C_4, g_c)}$ is greater than 1.*

Proof. The systole of (T, g_c) is equal to $\inf\{4a, 2\pi \cos(a\sqrt{2})\}$. By proposition 8 we know that $d(m, \tau^2(m)) \geq \pi$ ($\tau^2 = \sigma$), the proof is reduced to find a "good" lower bound of τ . Using the triangular inequality in (T, g_c) , we have

$$d(m, \tau^2(m)) \leq d(m, \tau(m)) + d(\tau(m), \tau^2(m))$$

but $d(p, \tau(p)) = d(\tau(p), \tau^2(p))$ since τ is an isometry of (T, g_c) . Then $d(m, \tau(m)) \geq \pi/2$, and the equality is achieved for the points m of the rotation axis. Note that using the same method,

we get a good lower bound of $\tau^3 = \tau^{-1}$.

Finally for $a = \pi/8$ the systole of (C_4, g_c) is equal to $\pi/2$. The volume is equal to

$$4 \int_0^{\frac{\pi}{2}} \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \cos \sqrt{x^2 + y^2} dx dy dz$$

Using Maple, we find the systolic ratio of our manifold, it is approximately equal to $1,05 > 1$. ■

6.2 Type C_6

In the flat case, the volume is equal to $\frac{1}{6} \det(a_1, a_2) |a_3|$ and the systole is equal to $\inf\{|a_3|/6, s\}$, where s is the systole of the flat 2-dimensional torus defined by the lattice of basis a_1, a_2 . Considering the usual normalisation $\frac{1}{6} |a_3| = 1$, the systolic ratio is equal to

$$\frac{s^3}{\det(a_1, a_2)} \quad \text{if } s \leq 1 \text{ and } \frac{1}{\det(a_1, a_2)} \quad \text{if } s \geq 1,$$

It is smaller than $2/\sqrt{3}$.

Now to improve this systolic ratio, we will start this time with the hexagonal torus (T, g_{hex}) defined in 4, since the lattice Δ should be hexagonal. To get the manifold C_6 , we take the quotient of (T, g_{hex}) by the subgroup generated by the isometry ϕ which sends a point (p, z) to the point $(r_{\pi/3}(p), z + \pi/3)$, the result is the manifold (C_6, g_{hex}) .

The manifold C_6 too can be seen in two different ways starting with the isometry $t_{a_3/4} \circ r_{a_3, \pi/2}$. This last one gives, when we go to the quotient manifold, a fixed point free isometry of order 6 (resp of order 3) of the torus (T, g_{hex}) (resp of (C_2, g_{hex})).

The transformations ϕ and ϕ^{-1} are of order 6 in (T, g_{hex}) and keep only one geodesic globally invariant.

The transformations ϕ^2 and ϕ^4 are of order 3 in (T, g_{hex}) and keep, in addition to the one of ϕ , 2 vertical geodesics globally invariant.

The transformation ϕ^3 is of order 2 and keeps, in addition to the one kept invariant by ϕ , 3 vertical geodesics globally invariant (see fig.3).

Theorem 3. *If the real number $a = \pi/12$, the systolic ratio $\frac{Sys(C_6, g_{hex})^3}{Vol(C_6, g_{hex})}$ is greater than $2/\sqrt{3}$.*

Proof. If $a = \pi/12$ we know, by lemma 2, that the systole of (T, g_{hex}) is equal to $\pi/3$. We also know, by Lemma 4, that the distance in (T, g_{hex}) between a point m and its image by ϕ^3 is greater or equal to π . Using the triangular inequality in (T, g_{hex}) we get

$$d(m, \phi^3(m)) \leq d(m, \phi(m)) + d(\phi(m), \phi^2(m)) + d(\phi^2(m), \phi^3(m))$$

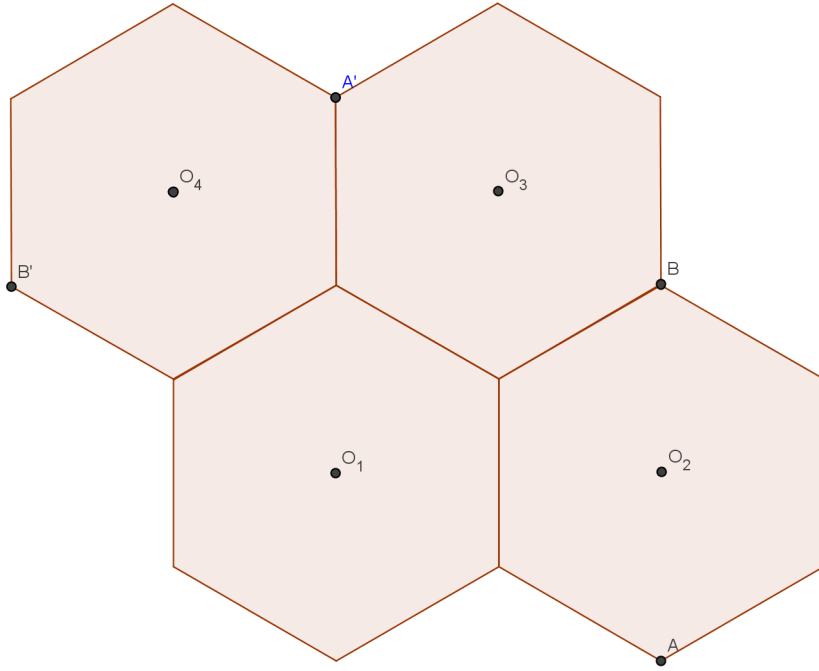


Figure 3: The transformations ϕ and ϕ^{-1} only keep fixed the vertical axis going through the point O_1 . The transformations ϕ^2 and ϕ^{-2} keep fixed, in addition to the axis going through O_1 , the vertical axes going through the points A et B . The transformation ϕ^3 keeps fixed, in addition to these three axes, the vertical ones going through the points O_i , ($i = 2, 3, 4$).

and then $d(m, \phi(m)) \geq \pi/3$ ($\phi^3 = \sigma$). Moreover, the distance in (T, g_{hex}) between a point m of coordinate (x, y, z) and a point m' of coordinate $(x', y', z + 2\pi/3)$ is greater than $\pi/3$. Indeed, if γ is a curve from m to m' we have

$$l(\gamma) = \int \sqrt{x'^2 + y'^2 + \psi(x, y)z'^2} dt \geq \int \sqrt{\psi(x, y)z'^2} \geq \cos \frac{2a}{\sqrt{3}} 2\pi/3 \geq \pi/3$$

The curves from $T_3(m)$ to m are too much long, and then for any m we have

$$\text{dist}_{(T, g_{hex})}(m, \phi^2(m)) \geq \pi/3$$

Finally, we conclude that $Sys(C_6, g_{hex}) = \pi/3$. The volume is equal to

$$\int_0^{\pi/3} \iint_D \cos \sqrt{x^2 + y^2} dy dx dz$$

With an approximation on "Maple" we calculate the systolic ratio $\frac{Sys^3(C_6, g_{hex})}{Vol(C_6, g_{hex})} \simeq 1, 18$. ■

6.3 Type $C_{2,2}$

It is the easiest case since the systolic geodesics of the best metric among the flat ones are isolated.

In the flat case, the systole is equal to $\inf\{a_1/2, a_2/2, a_3/2\}$. The volume is equal to $\frac{|a_1||a_2||a_3|}{4}$. The systolic ratio is smaller than $1/2$, the equality is achieved if and only if $|a_1| = |a_2| = |a_3|$. In that case the systolic geodesics are isolated and so they do not cover the manifold $C_{2,2}$.

The criterion seen in the introduction allows us to conclude that the flat metric on $C_{2,2}$ are not the best for the isosystolic inequality.

6.4 Type C_3

In the flat case, the volume is equal to $\frac{1}{3} \det(a_1, a_2) |a_3|$ but also to $\frac{\sqrt{3}}{6} |a_1| |a_3|$, and the systole is equal to $\inf\{|a_3|/3, |a_1|\}$, since the lattice generated by a_1 and a_2 is hexagonal. We conclude easily that the systolic ratio is less or equal to $2/\sqrt{3}$. The equality is achieved for $|a_3| = 3|a_2| = 3|a_1|$.

To improve this systolic ratio, we start with the hexagonal torus (T, g_{hex}) defined in section 4, since the lattice Δ should be hexagonal. To get the manifold C_3 , we take the quotient of (T, g_{hex}) by the subgroup generated by the isometry φ which sends a point (p, z) to the point $(r_{2\pi/3}(p), z + 2\pi/3)$, the result is the manifold (C_3, g_{hex}) .

Since the manifold C_3 is not a quotient of C_2 , it does not contain surfaces that are Klein bottles or Möbius bands. Then, our previous methods of getting a lower bound for the systole cannot be applied. Though, a special and more general argument is necessary.

Let φ_c be the isometry of (T, g_{hex}) which sends (p, z) to the point $(r_{2\pi/3}(p), z + c)$. The quotient of (T, g_{hex}) by the subgroup generated by φ_c is clearly a manifold homeomorphic to C_3 , we

denote it by (C_3, g_{hex}^c) . Let γ be the vertical geodesic in a domain D_p in (C_3, g_{hex}^c) going through the point p , it has length equal to c . Now let H be a piecewise smooth variation of γ through geodesics joining a point m to $\varphi_c(m)$, we impose that these curves stay in D_p and do not touch the singularity.

Lemma 5. *The second variation of H at the curve γ is strictly positive if $0 < c < 2\pi/3$.*

Proof. Let O be a small tubular neighborhood of γ and let Ω be the set of geodesics in D_p from $m_t \in O$ to $\varphi_c(m_t)$ that do not touch the singularity (one parameter family since the situation is invariant under rotation around γ). Then

$$H :] - \epsilon, \epsilon[\longrightarrow \Omega$$

$$t \longrightarrow \gamma_t : [0, 1] \rightarrow (C_3, g_{hex}^c)$$

is such that $H(o) = \gamma$. Let $T = \frac{\partial \gamma_t}{\partial s}$ (velocity vector of γ_t), and $V = \frac{\partial \gamma}{\partial t}|_{\gamma_t}$ (the Jacobi field along γ), and set $L = \int_0^c |T| ds$. We have then

$$\frac{\partial L}{\partial t} = \frac{1}{L(t)} \int_0^c g_{hex}^c(V, \nabla_V T) ds$$

since $\nabla_T V - \nabla_V T = [V, T] = 0$ we get

$$L \frac{\partial L}{\partial t} = [g_{hex}^c(V, T)]_0^c \quad (\text{1st variation formula})$$

Now

$$\begin{aligned} \frac{\partial}{\partial t} \left(L \frac{\partial L}{\partial t} \right) &= \left(\frac{\partial L}{\partial t} \right)^2 + L \frac{\partial^2 L}{\partial t^2} \\ &= \int_0^c (|\nabla_T V|^2 + g_{hex}^c(T, \nabla_V \nabla_T V)) ds \\ &= \int_0^c |\nabla_T V|^2 + \int_0^c g_{hex}^c(T, \nabla_T \nabla_V V) + \int_0^c g_{hex}^c(R(V, T)V, T) \end{aligned}$$

where R is the curvature tensor of g_{hex}^c . Now since the curvature in the direction of the plane (T, V) is equal to 1 we get

$$\begin{aligned} L \frac{\partial^2 L}{\partial t^2} &= \int_0^c |\nabla_T V|^2 - L^2 \int_0^c |V|^2 \\ &= \int_0^c (\nabla_T g_{hex}^c(V, \nabla_T V) - g_{hex}^c(V, R(T, V)T)) - L^2 \int_0^c |V|^2 = g_{hex}^c(V, \nabla_T V)|_0^c \end{aligned}$$

(see [Che75] p. 20 for more details on the second variation formula).

Now V is a Jacobi Field orthogonal to γ and so can be written in the form $V = f_1 E_1 + f_2 E_2$, where (E_1, E_2) is an orthonormal basis of the (horizontal) plane and parallel along γ . We can suppose that $V(0) = E_1$ and $V(c) = E_1 \cos(2\pi/3) + E_2 \sin(2\pi/3)$. Now solving the Jacobi Field equation $V'' + V = 0$ we get $f_1(s) = \cos(s) + \frac{\cos(2\pi/3) - \cos(c)}{\sin(c)} \sin(s)$ and $f_2(s) = \frac{\sin(2\pi/3)}{\sin(c)} \sin(s)$.

Finally

$$\begin{aligned} g_{hex}^c(V, \nabla_T V)|_0^c &= f_2(c)f_2'(c) + f_1(c)f_1'(c) - f_1'(0) \\ &= \sin^2(2\pi/3)(\cos(c) - \cos(2\pi/3)) + \cos(2\pi/3)(\cos^2(c) - \cos^2(2\pi/3)) \end{aligned}$$

■

Remark 9. This lemma shows that there exists a neighbourhood U of the geodesic γ in which γ is of minimum length among the geodesics joining any point m to $\varphi_c(m)$.

Theorem 4. If $c = 2\pi/3$ and $a = \pi/6$, the systolic ratio $\frac{Sys(C_3, g_{hex})^3}{Vol(C_3, g_{hex})}$ is greater than $2/\sqrt{3}$.

Proof. We consider in the neighborhood U a hexagon H "parallel" (c.f. remark 7) to the boundary ∂D_p . Let δ be a curve in (\mathbb{R}^3, g_{hex}) from a point m in D_p to $\varphi_c(m)$, the minimal projection of δ on the boundary ∂H gives a curve δ' in U joining two conjugate points by the transformation φ_c , then we have by lemma 3 and remark 7

$$l(\delta) \geq l(\delta') \geq c$$

The same arguments as the ones used in section 5 show that $d_{(T, g_{hex})}(m, \varphi_c(m)) \geq c$.

Now passing to the limit when $c \rightarrow 2\pi/3$ we get

$$d_{(T, g_{hex})}(m, \varphi(m)) \geq 2\pi/3$$

This allows us to calculate the systole of (C_3, g_{hex}) , when $a = \pi/6$. It is equal to $2\pi/3$ (of course we use Lemma 2 too). The volume is equal to

$$\int_0^{\frac{2\pi}{3}} \iint_D \cos \sqrt{x^2 + y^2} dy dx dz$$

As before we calculate this integral using Maple, and we get an approximation of $\frac{Sys(C_3, g_{hex})^3}{Vol(C_3, g_{hex})} \simeq 1.24$.

■

Remark 10. The previous proof is also valid for the manifolds (C_6, g_{hex}) and (C_4, g_c) , and allows us to find the good lower bound of their systoles. But the method used in section 6 is a lot more simple (we just used the triangular inequality), this is due to the existence of Klein bottles and Möbius bands in these manifolds.

7 Comparison between (C_2, g_{hex}) and flat hexagonal 3-dimensional torus

Among flat tori of dimension 3, the hexagonal one is the best for the isosystolic inequality. It is the quotient of \mathbb{R}^3 by the lattice that has a basis (a_1, a_2, a_3) such that $(a_i, a_j) = \pi/3$ for $i \neq j$. It is known that this torus, that we denote by T_{hex}^3 , is a very good candidate to realize the systolic constant of tori of dimension 3, it satisfies the following properties:

- At any point in T_{hex}^3 there exists exactly 6 systolic geodesics going through the point.
- The systolic geodesics of any systolic class of T_{hex}^3 cover the torus. A systolic class is an element of the fundamental group that contains at least one systolic geodesic.

Our singular metric (C_2, g_{hex}) verifies the second property and a stronger one than the first: At any point outside the singularity of (C_2, g_{hex}) , there exists infinitely many systolic geodesics going through the point.

For the points on the singularity, there are 5 systolic geodesics going through any of these points: 3 in the horizontal flat 2-torus and 2 in the surface $\theta = \text{constant}$. The number of systolic geodesics going through the points of the singularity is less than the case of T_{hex}^3 , but this does not cause any trouble since the singularity has zero measure.

We think that as for the 3-dimensional hexagonal torus the manifold (C_2, g_{hex}) is a very good candidate to realize the systolic constant because it has an abundance of systolic geodesics that can be seen by the fact that it satisfies the properties mentioned above.

When speaking about the metrics (C_3, g_{hex}) , (C_6, g_{hex}) and (C_4, g_c) , they still satisfy the property of being covered by systolic geodesics mentioned in the introduction. But we cannot say if they satisfy something stronger as for the manifold (C_2, g_{hex}) since we do not have much information about the length of vertical geodesics.

The following table allows to do the comparison between the biggest systolic ratio of the flat metrics ($\tau(\text{flat})$) and the biggest ones of the singular metrics that we have constructed in this paper ($\tau(\text{singular})$) on the orientable Bieberbach 3-manifolds of type C_2, C_3, C_4 and C_6 .

type	$\tau(\text{flat})$	approximate value	$\tau(\text{singular})$
C_2	$\frac{2}{\sqrt{3}}$	$\approx 1,154$	$\approx 1,38$
C_3	$\frac{2}{\sqrt{3}}$	$\approx 1,154$	$\approx 1,24$
C_4	1	1	$\approx 1,05$
C_6	$\frac{2}{\sqrt{3}}$	$\approx 1,154$	$\approx 1,18$

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